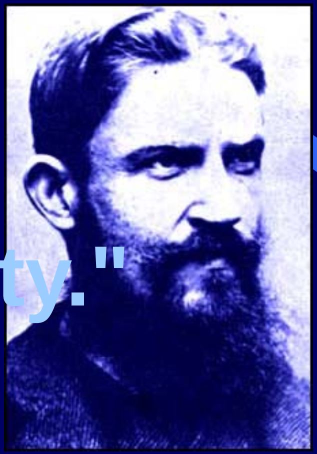
Introduction to Fuzzy Inference

Why Fuzzy?

The Wisdom of Experience ... ???

- "(Fuzzy theory's) delayed exploitation outside Japan teaches several lessons. ...(One is) the traditional intellectualism in engineering research in general and the *cult of analyticity* within control system engineering research in particular."

  E.H. Mamdami, 1975 father of fuzzy control (1993).



- "All progress means war with society."

  George Bernard Shaw

Bipolar Paradoxes

- “ never tell the truth”
-
-
-

Crisp Versus Fuzzy

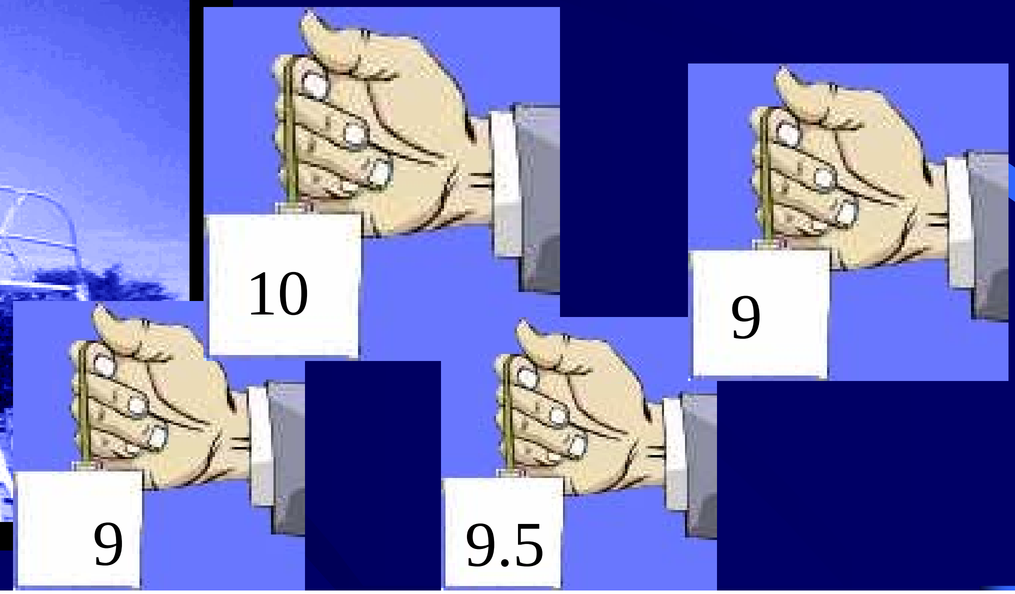
- Conventional or *crisp* sets are binary. An element either belongs to the set or doesn't.
- *Fuzzy sets*, on the other hand, have grades of memberships. The set of cities `far' from Los Angeles is an example.

$$\mu_{LA} = 0.0 / \mathrm{LA} + 0.5 / \mathrm{Chicago}$$
$$+ 0.8 / \mathrm{New\ York} + 0.9 / \mathrm{London}$$

Fuzzy Linguistic Variables

- The term *far* used to define this set is a *fuzzy linguistic variable*.
- Other examples include *close*, *heavy*, *light*, *big*, *small*, *smart*, *fast*, *slow*, *hot*, *cold*, *tall* and *short*.

***e.g.* On a scale of one to 10, how *good* was the dive?**



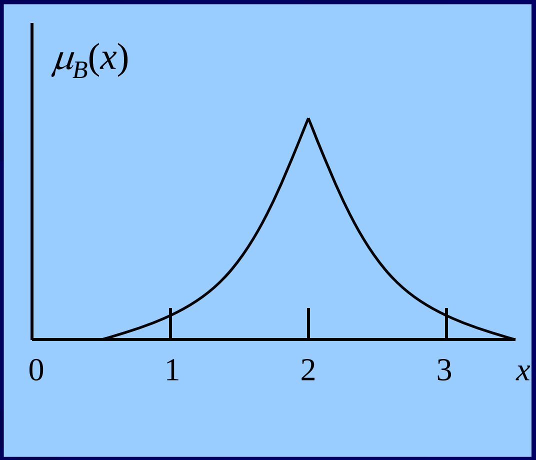
Continuous Fuzzy Membership Functions

- The set, *B*, of numbers *near* to two is

$$\mu_B(x) = \frac{1}{(x-2)^2}$$

- or...

$$\mu_B(x) = e^{-|x-2|}$$



Fuzzy Subsets

- A fuzzy set, *A*, is said to be a subset of *B* if

$$\mu_A(x) \le \mu_B(x)$$

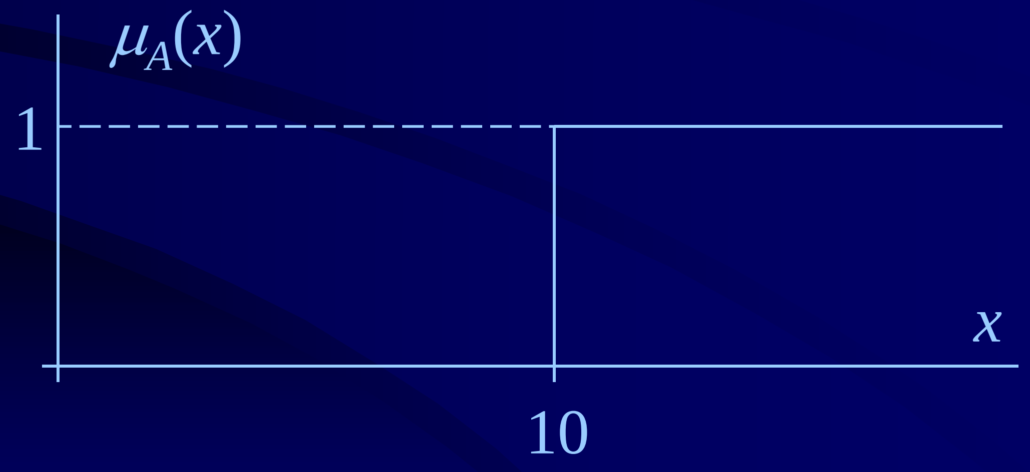
- *e.g. B* = far and *A*=*very* far.
- For example...

$$\mu_A(x) = \mu_B^2(x)$$

Crisp Membership Functions

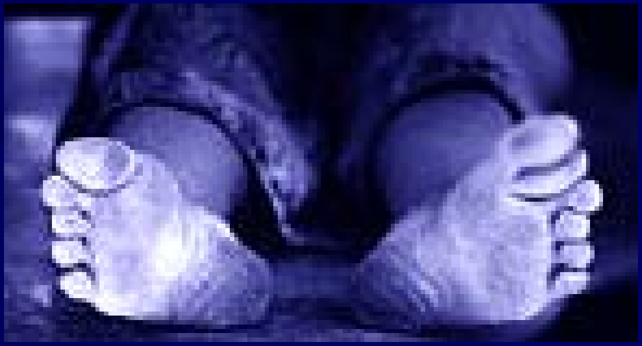
- Crisp membership functions are either one or zero.
- e.g. Numbers greater than 10.

$A = \{x \mid x > 10\}$



Fuzzy Versus Probability

- Fuzzy ≠ Probability
- Example #1
  – Billy has ten toes. The *probability* Billy has nine toes is zero. The fuzzy membership of Billy in the set of people with nine toes, however, is nonzero.



Fuzzy Versus Probability

Example #2

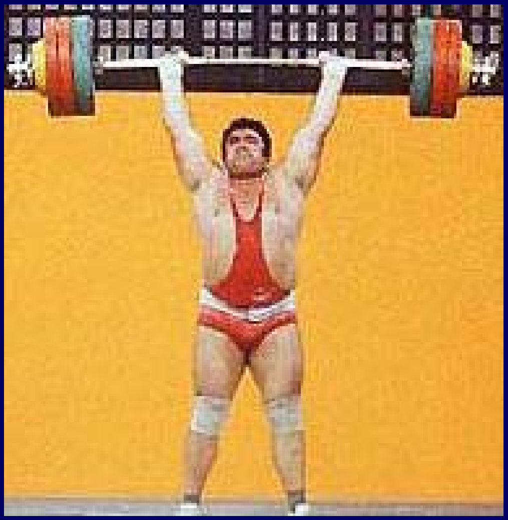
- A bottle of liquid has a probability of ½ of being rat poison and ½ of being pure water.
- A second bottle's contents, in the fuzzy set of liquids containing *lots* of rat poison, is ½.
- The meaning of ½ for the two bottles clearly differs significantly and would impact your choice should you be dying of thirst.

(cite: Bezdek)

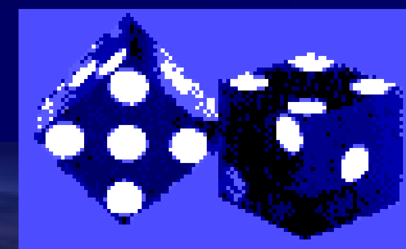
Fuzzy Versus Probability

Example #3

- Fuzzy is said to measure “possibility” rather than “probability”.
- Difference
  - All things possible are not probable.
  - All things probable are possible.
- Contrapositive
  - All things impossible are improbable
  - Not all things improbable are impossible



Fuzzy Vs. Crisp Probability



- The probability that a fair die will show six is 1/6. This is a crisp probability. All credible mathematicians will agree on this exact number.
- The weatherman's forecast of a probability of rain tomorrow being 70% is also a fuzzy probability. Using the same meteorological data, another weatherman will typically announce a different probability.

Fuzzy Logic

Criteria for fuzzy "and", "or", and "complement"

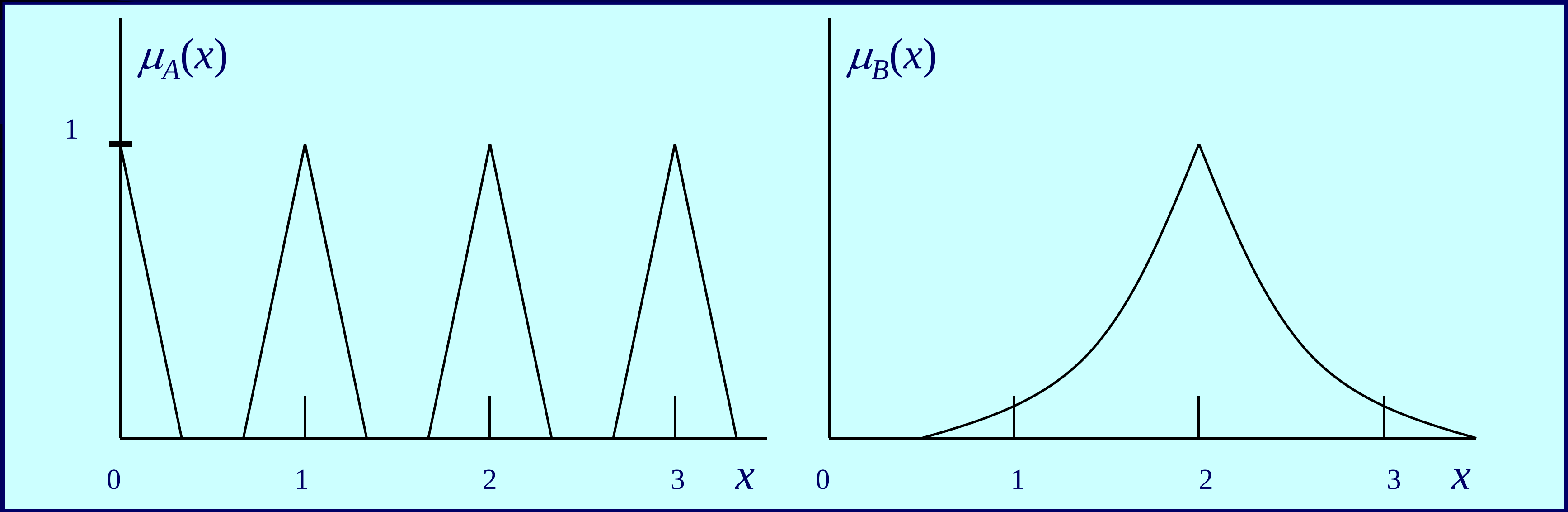
- Must meet crisp boundary conditions
- Commutative
- Associative
- Idempotent
- Monotonic

Fuzzy Logic

Example Fuzzy Sets to Aggregate...

$A = \{ x \mid x \text{ is } \textit{near} \text{ an integer}\}$

$B = \{ x \mid x \text{ is } \textit{close} \text{ to } 2\}$

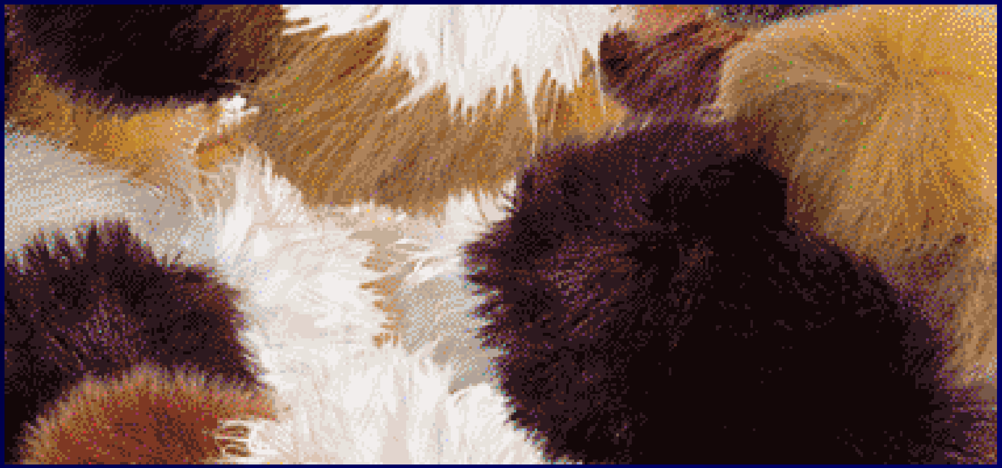


Fuzzy Union

- Fuzzy Union (logic “or”)

$$\mu_{A+B}(x) = \max[\mu_{\mathrm{A}}(x), \mu_B(x)]$$

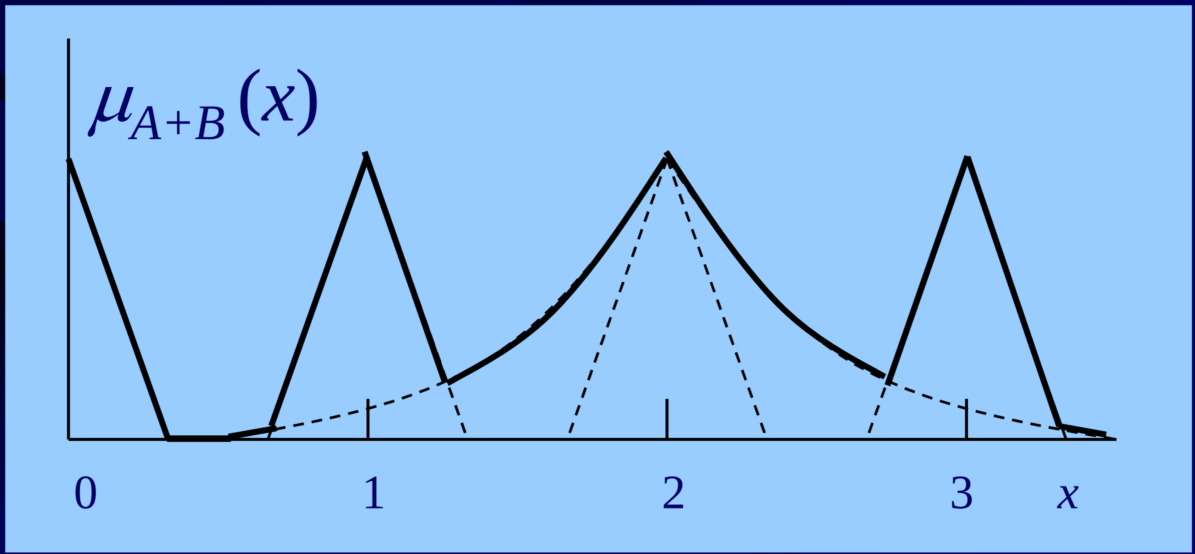
  - Meets crisp boundary conditions
  - Commutative
  - Associative
  - Idempotent
  - Monotonic



Fuzzy Union

A OR B = $A+B$ ={ x | (x is *near* an integer) OR (x is *close* to 2)}

$$= \text{MAX}\left[\mu_A(x), \mu_B(x)\right]$$

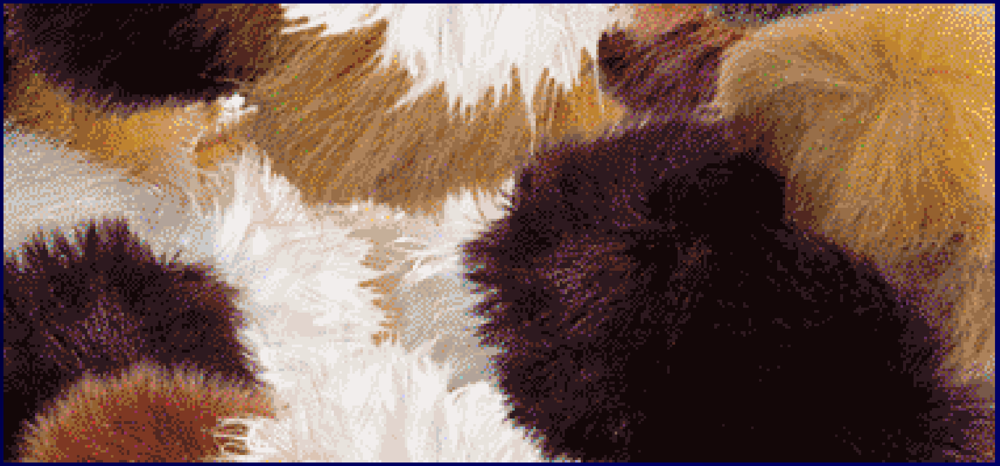


Fuzzy Intersection

- Fuzzy Intersection (logic "and")

$$\mu_{A \bullet B}(x) = \min\left[\mu_{A}(x), \mu_{B}(x)\right]$$

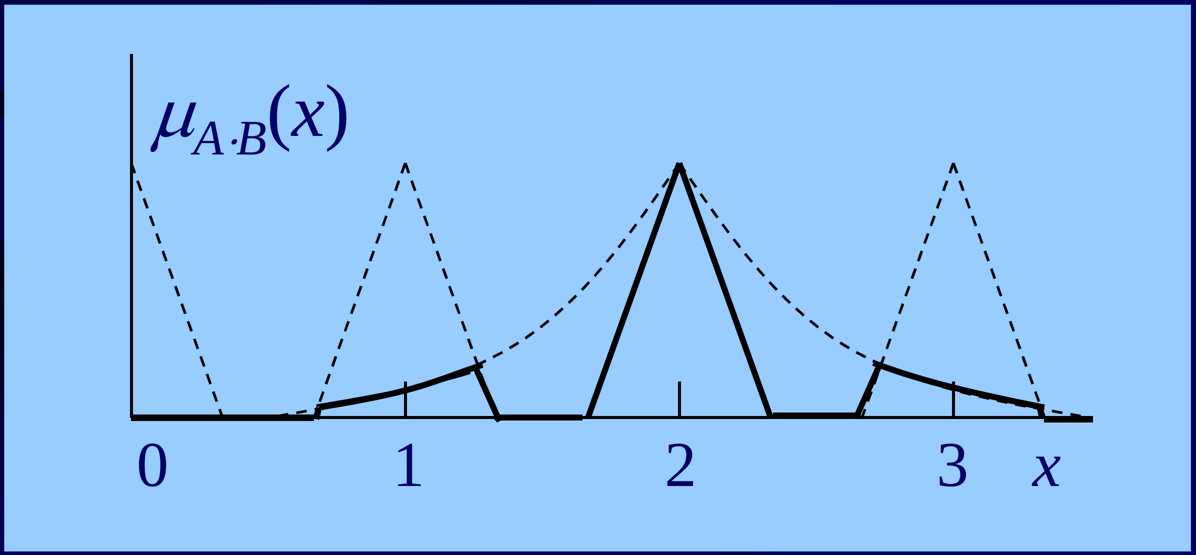
- Meets crisp boundary conditions
- Commutative
- Associative
- Idempotent
- Monotonic



Fuzzy Intersection

A AND $B = A{\cdot}B =\{\ x \mid (x$ is *near* an integer) AND (x is *close* to 2)$\}$

$= \text{MIN}\left[\mu_A(x),\ \mu_B(x)\right]$

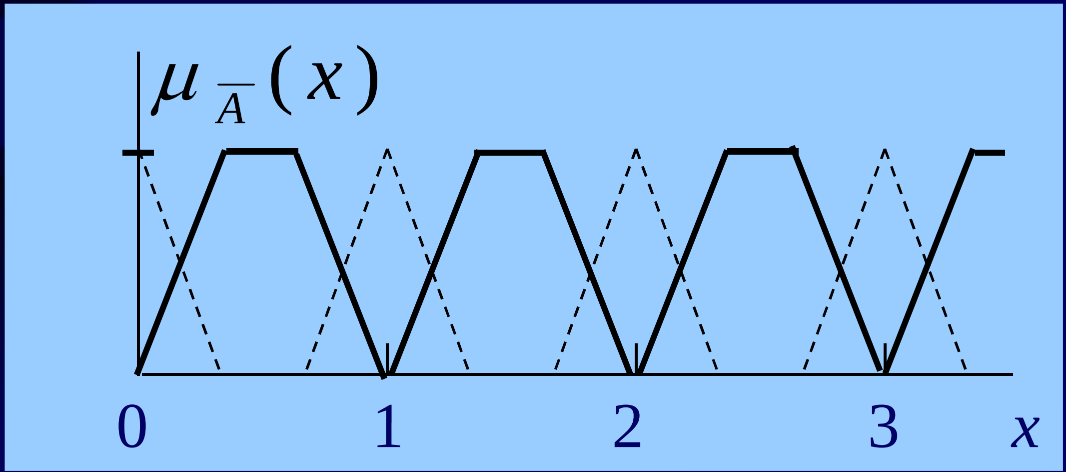


Fuzzy Complement

The complement of a fuzzy set has a membership function...

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

complement *of A* ={ *x* | *x* is <u>not</u> *near* an integer}



Associativity

Min-Max fuzzy logic has intersection distributive over union...

$$\mu_{A\bullet(B+C)}(x) = \mu_{(A+B)\bullet(A+C)}(x)$$

since

min[A,max(B,C)]=min[max(A,B), max(A,C)]

Associativity

Min-Max fuzzy logic has union distributive over intersection...

$$\mu_{A+(B\bullet C)}(x) = \mu_{(A\bullet B)+(A\bullet C)}(x)$$

since

max[A,min(B,C)]= max[min(A,B), min(A,C)]

DeMorgan's Laws

Min-Max fuzzy logic obeys DeMorgans Law #1...

$$\mu_{\overline{B \bullet C}}(x) = \mu_{\overline{B}+\overline{C}}(x)$$

since

1 - min(*B*,*C*)= max[(1-*A*), (1-*B*)]

DeMorgan's Laws

Min-Max fuzzy logic obeys DeMorgans Law #2...

$$\mu_{\overline{B+C}}(x) = \mu_{\overline{B}\bullet\overline{C}}(x)$$

since

1 - max(B,C)= min[(1-A), (1-B)]

Excluded Middle

Min-Max fuzzy logic fails ***The Law of Excluded Middle.***

$$A \cdot \overline{A} \neq \phi$$

since

$$\min(\mu_A, 1-\mu_A) \neq 0$$

Thus, (the set of numbers *close* to 2) AND (the set of numbers <u>not</u> *close* to 2) ≠ null set

Contradiction

Min-Max fuzzy logic fails the ***The Law of Contradiction.***

$$A + \overline{A} \neq \mathrm{U}$$

since

$$\max(\mu_A, 1-\mu_A) \neq 1$$

Thus, (the set of numbers *close* to 2) OR (the set of numbers <u>not</u> *close* to 2) ≠ universal set

Other Fuzzy Logics

There are numerous other operations OTHER than Min and Max for performing fuzzy logic intersection and union operations.

A common set operations is ***sum-product inferencing***, where…

$$\mu_{A\bullet B}(x) = \mu_{\mathrm{A}}(x)\mu_B(x)$$

$$\mu_{A+B}(x) = \min[\mu_{\mathrm{A}}(x) + \mu_B(x), 1]$$

Cartesian Product

- The intersection and union operations can also be used to assign memberships on the Cartesian product of two sets.
- Consider, as an example, the fuzzy membership of a set, *G*, of liquids that taste *good* and the set, *LA*, of cities close to Los Angeles

$$\mu_G = 0.0/\text{Swamp Water} + 0.5/\text{Radish Juice} + 0.9/\text{Grape Juice}$$

$$\mu_{LA} = 0.0/\text{LA} + 0.5/\text{Chicago} + 0.8/\text{New York} + 0.9/\text{London}$$

Cartesian Product

- We form the set...

$$E = G \cdot LA$$

= liquids that taste *good AND* cities that are *close* to LA

- The following table results...

	LosAngeles(0.0)	Chicago (0.5)	New York (0.8)	London(0.9)
Swamp Water (0.0)	0.00	0.00	0.00	0.00
Radish Juice (0.5)	0.00	0.25	0.40	0.45
Grape Juice (0.9)	0.00	0.45	0.72	0.81

Fuzzy Inference

Crisp input

↓

Fuzzification

“antecedent”

↓

Rules

“consequent”

↓

Defuzzification

↓

Crisp Output Result

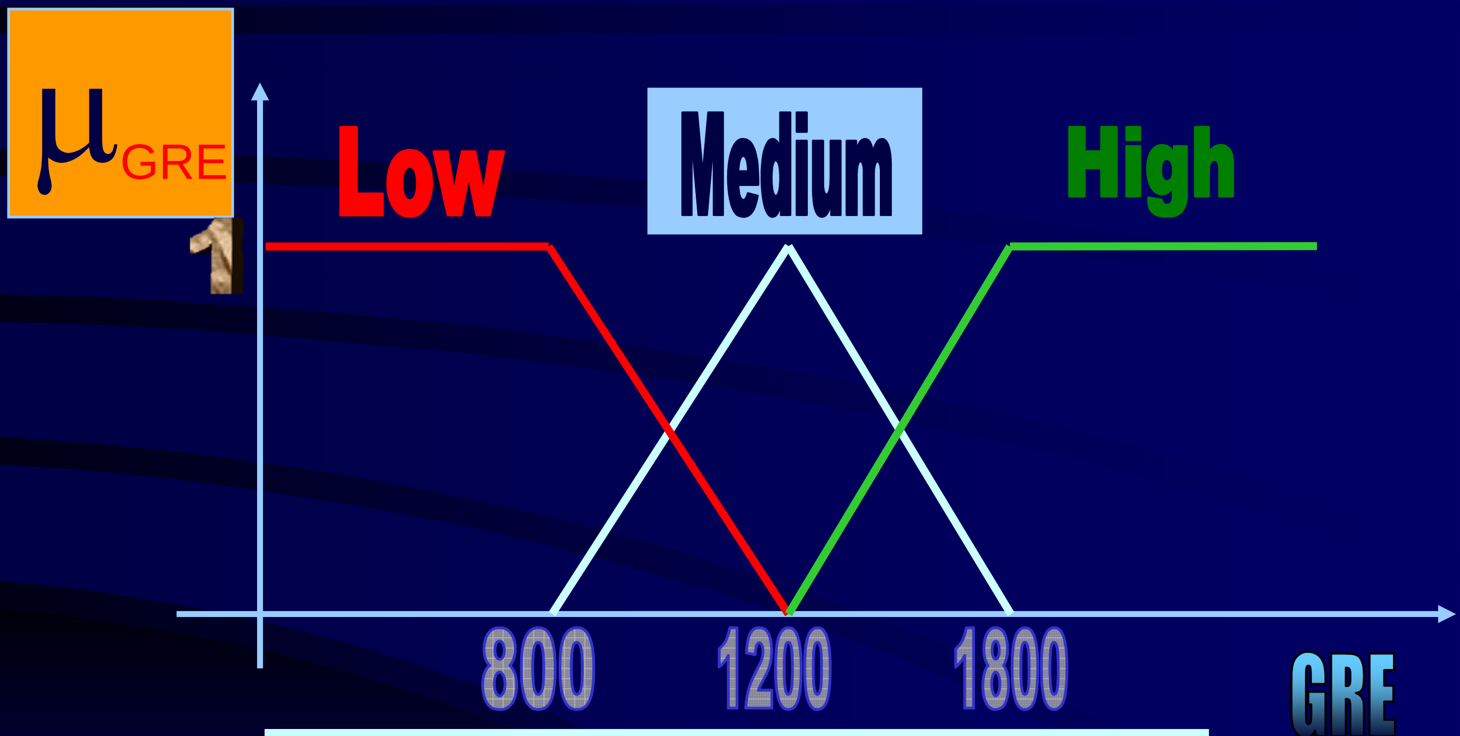
Fuzzy Inference Example

Fuzzy Rule Table

		GRE		
		H	M	L
GPA	H	E	VG	F
	M	G	G	B
	L	F	B	B

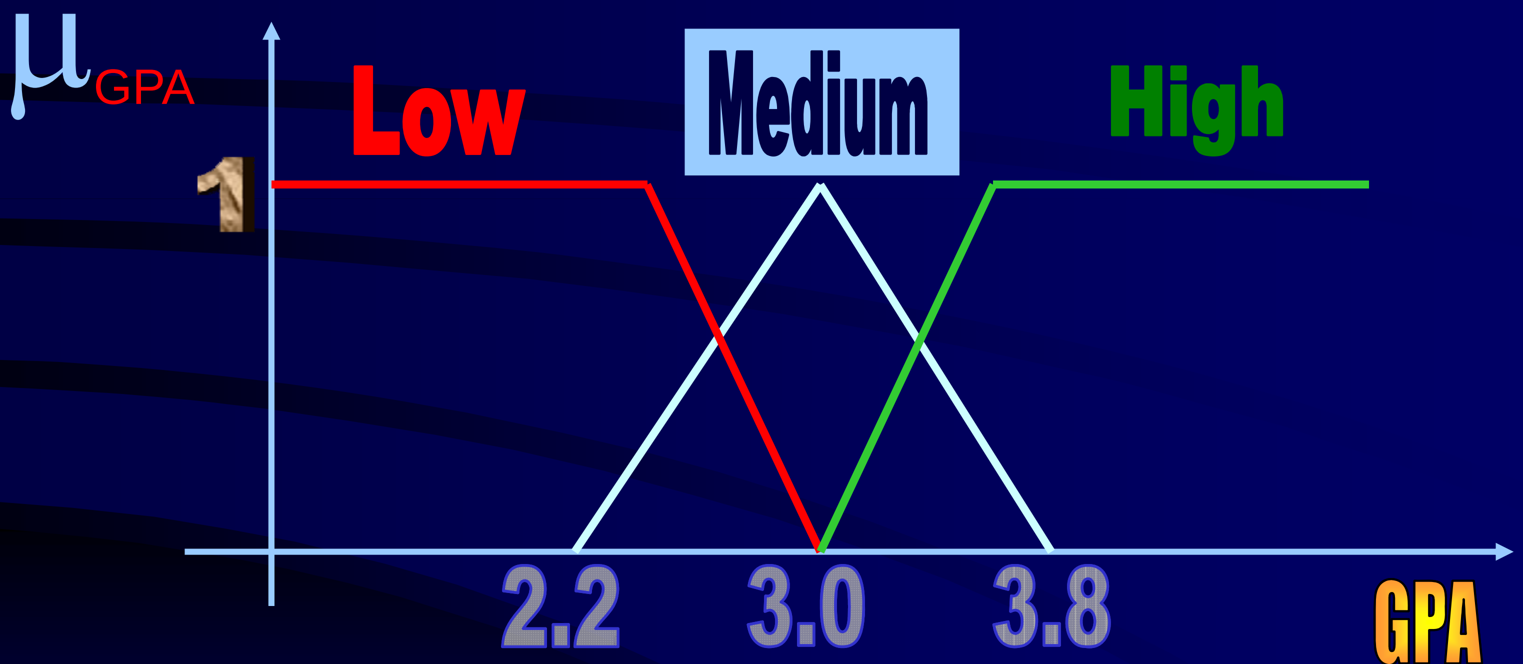
Fuzzification

Membership Functions for GRE



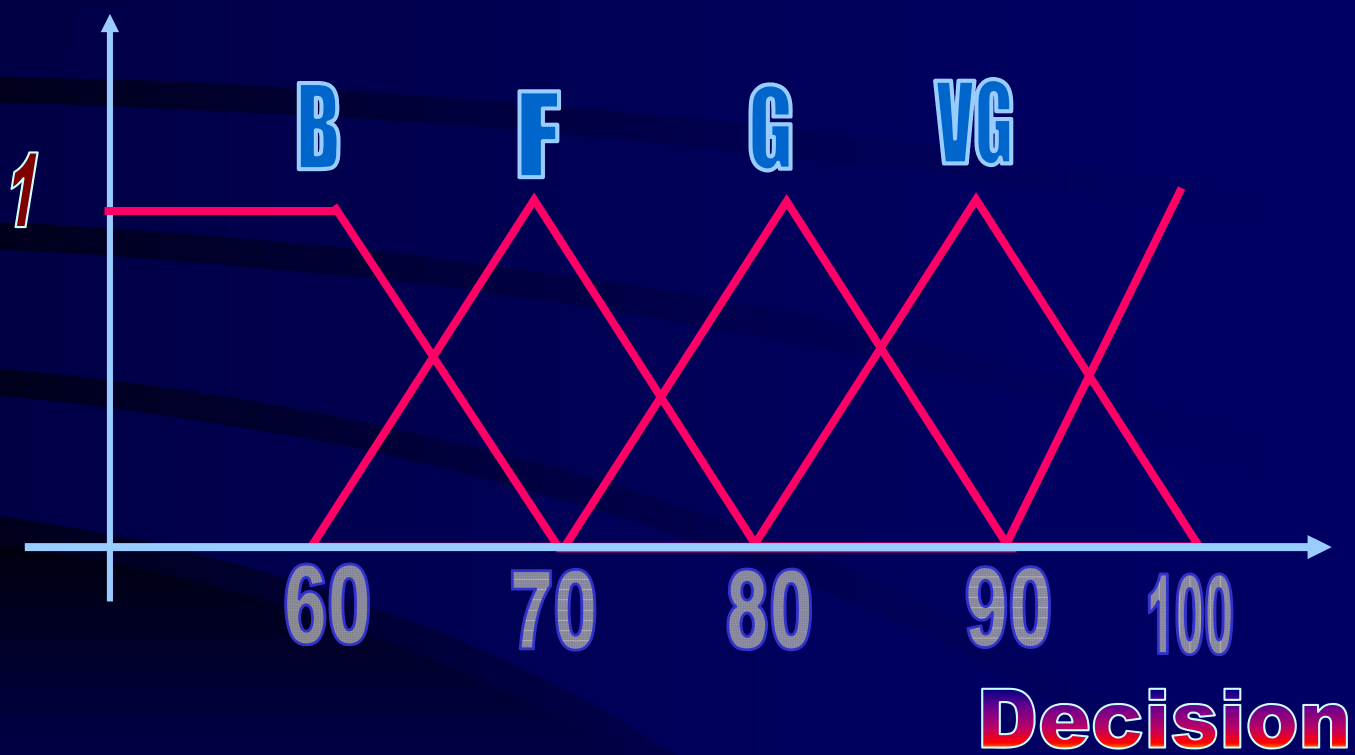
$$\mu_{GRE} = \{\mu_L , \mu_M , \mu_H \}$$

Membership Functions for the GPA



$$\mu_{GPA} = \{\mu_{L}, \mu_{M}, \mu_{H}\}$$

Membership Function for the Consequent



Fuzzification

-
-

$$\mu_{GRE} = \{\mu_L = 0.8\ ,\ \mu_M = 0.2\ ,\ \mu_H = 0\}$$

$$\mu_{GPA} = \{\mu_L = 0\ ,\ \mu_M = 0.6\ ,\ \mu_H = 0.4\}$$

Activated Rules

		GRE		
		H	M	L
GPA	H	E	VG	F
	M	G	G	B
	L	F	B	B

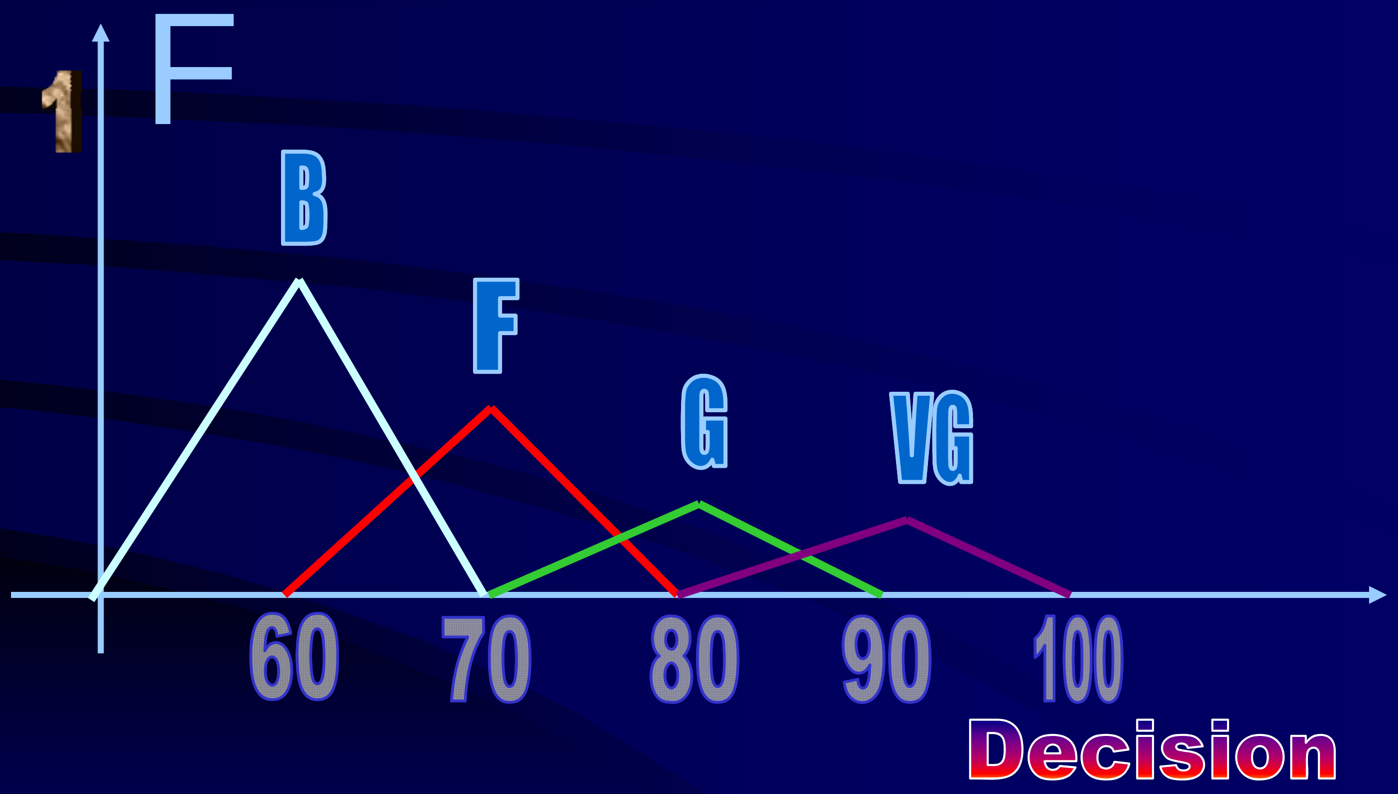
Memberships of Activated Rules

GPA \ GRE	0	0.2	0.8
0.4	0	0.2	0.4
0.6	0	0.2	0.6
0	0	0	0

F = {B,F,G,VG,E}

F = {0.6, 0.4, 0.2, 0.2, 0}

Weight Consequent Memberships

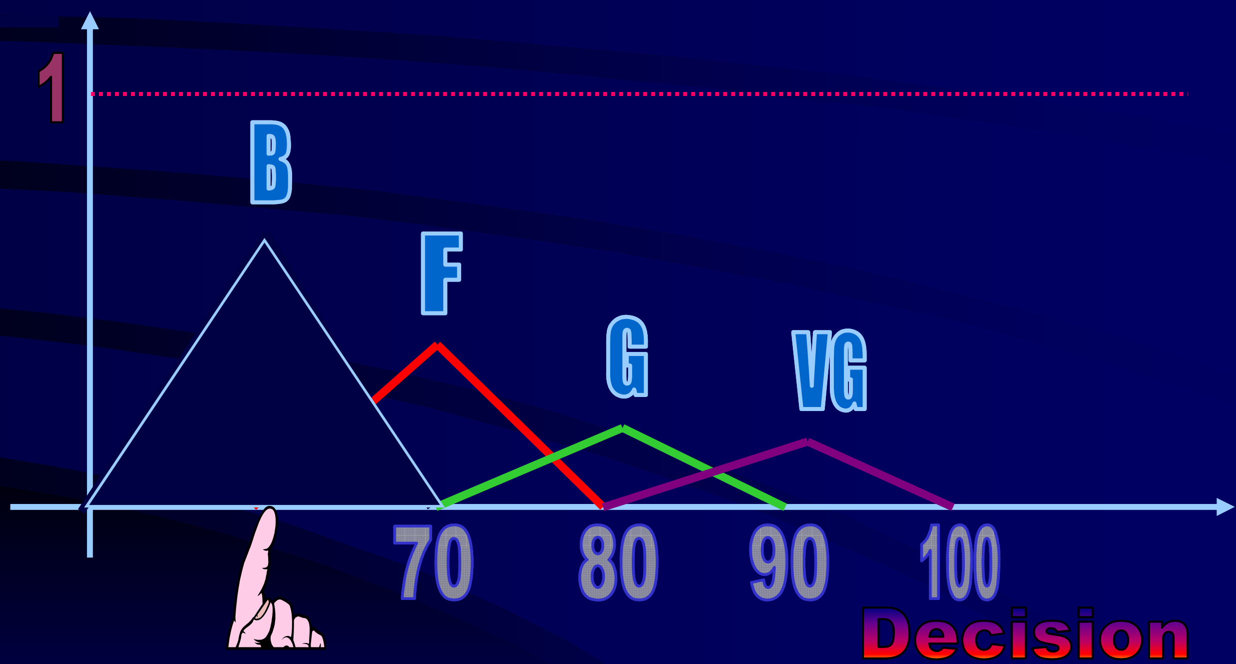


Defuzzification

-
- F

Max Method

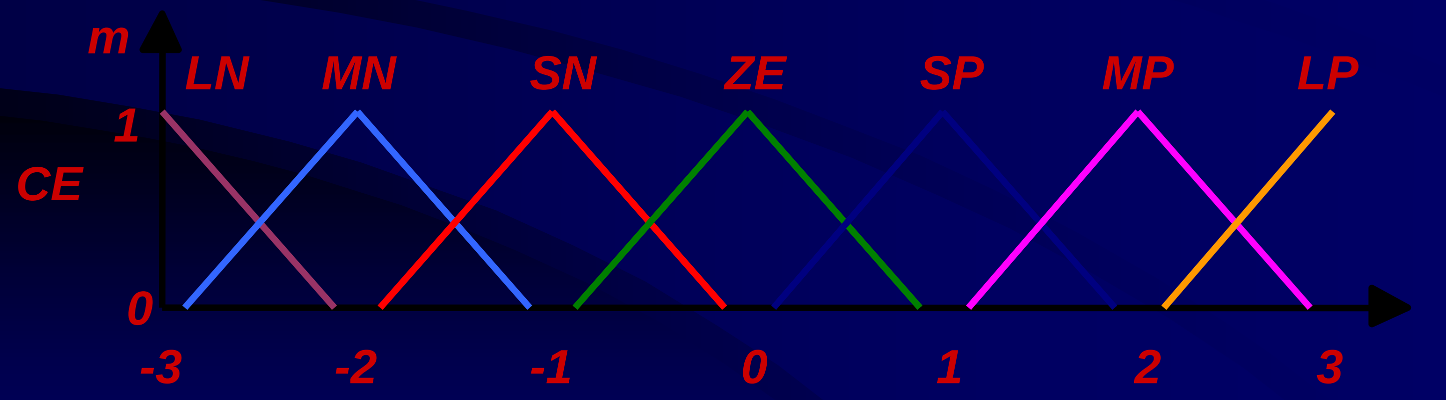
Decision: Max Method



Example: Fuzzy Table for Control

		CE						
		LN	MN	SN	ZE	SP	MP	LP
	LN	LN	LN	LN	LN	MN	SN	SN
	MN	LN	LN	LN	MN	SN	ZE	ZE
	SN	LN	LN	MN	SN	ZE	ZE	SP
E	ZE	LN	MN	SN	ZE	SP	MP	LP
	SP	SN	ZE	ZE	SP	MP	LP	LP
	MP	ZE	ZE	SP	MP	LP	LP	LP
	LP	SP	SP	MP	LP	LP	LP	LP

Membership Functions



Rule Aggregation

		CE						
		LN	MN	SN	ZE	SP	MP	LP
E	LN	LN	LN	LN	LN	MN	e. SN 0.2	f. SN 0.0
	MN	LN	LN	LN	MN	d. SN 0.5	ZE	ZE
	SN	LN	LN	MN	c.SN 0.3	ZE	ZE	SP
	ZE	LN	MN	b.SN 0.4	ZE	SP	MP	LP
	SP	a. SN 0.1	ZE	ZE	SP	MP	LP	LP
	MP	ZE	SP	SP	MP	LP	LP	LP
	LP	SP	SP	MP	LP	LP	LP	LP

Consequent is or SN if *a* or *b* or *c* or *d* or *f*.

Rule Aggregation

Consequent is or SN if *a* or *b* or *c* or *d* or *f*.

Consequent Membership = max(a,b,c,d,e,f) = 0.5

More generally:

$$agg_{\alpha}(\vec{x}) = \left[\frac{1}{N} \sum_{n=1}^{N} x_n^{\alpha} \right]^{1/\alpha}$$

Rule Aggregation

$$agg_{\alpha}(\vec{x}) = \left[\frac{1}{N} \sum_{n=1}^{N} x_n^{\alpha} \right]^{1/\alpha}$$

Special Cases:

$$agg_{-\infty}(\vec{x}) = \min_n x_n \text{; minimum}$$

$$agg_{-1}(\vec{x}) = \left[\frac{1}{N} \sum_{n=1}^{N} \frac{1}{x_n} \right]^{-1} \text{; harmonic mean}$$

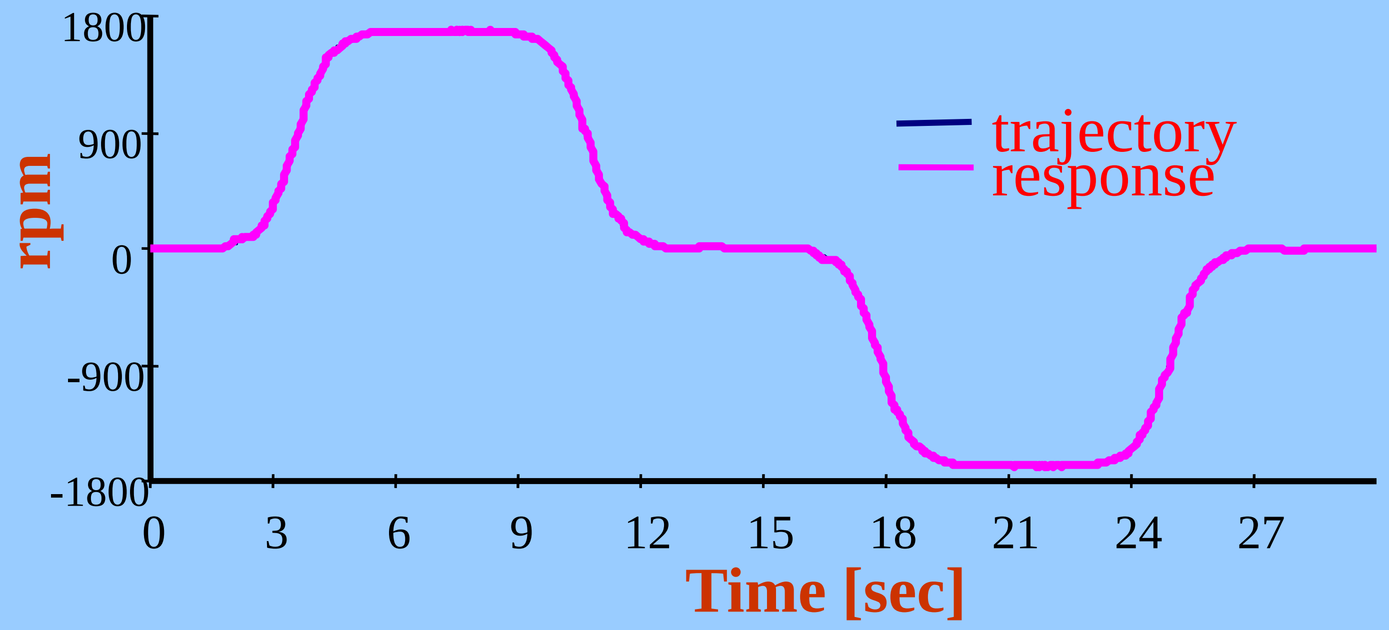
$$agg_{0}(\vec{x}) = \left[\Pi_{n-1}^{N} x_n \right]^{1/N} \text{; geometric mean}$$

$$agg_{1}(\vec{x}) = \frac{1}{N} \sum_{n=1}^{N} x_n \text{; average}$$

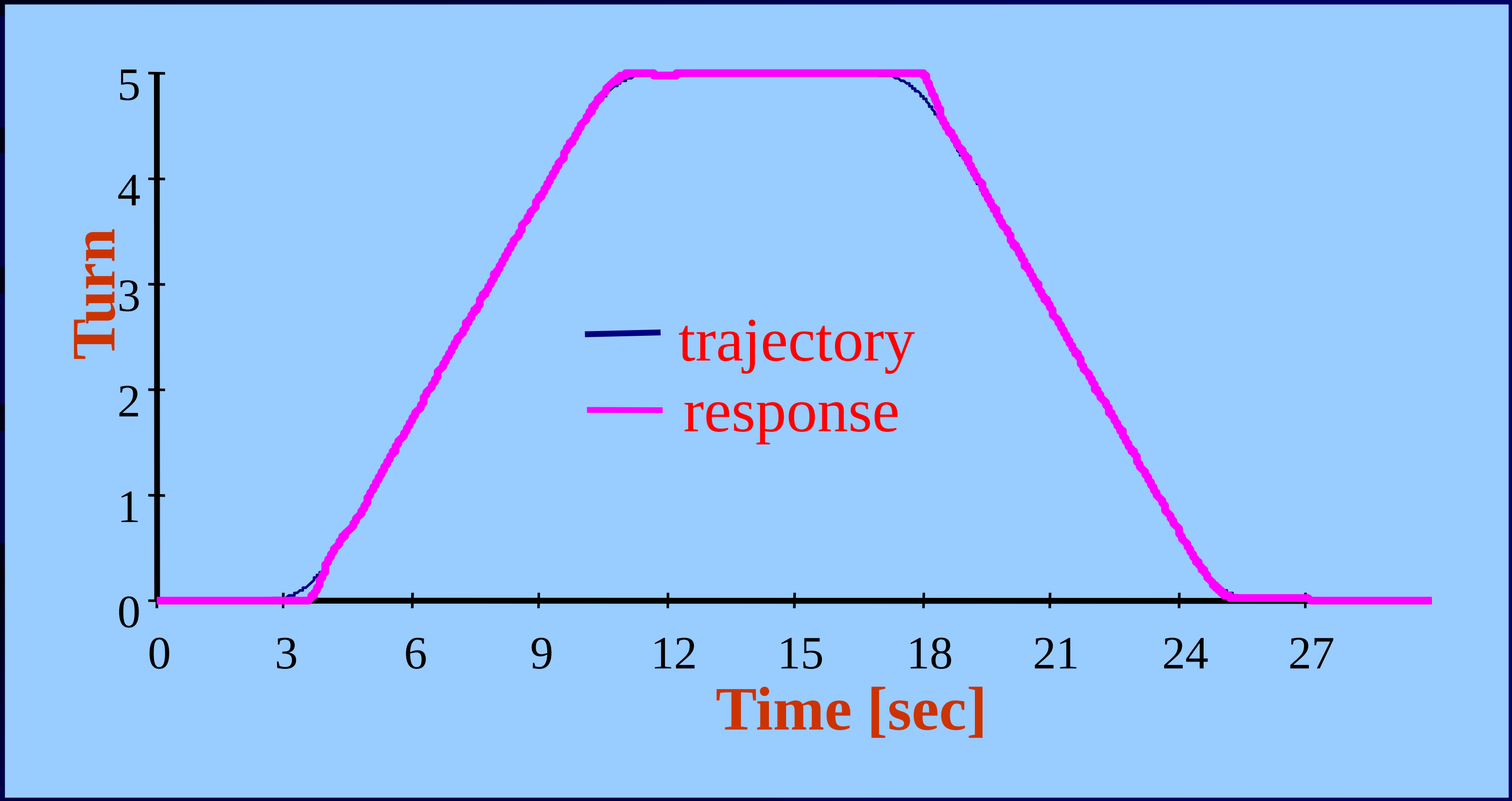
$$agg_{2}(\vec{x}) = \left[\frac{1}{N} \sum_{n=1}^{N} x_n^2 \right]^{1/2} \text{; rms}$$

$$agg_{\infty}(\vec{x}) = \max_n x_n \text{; maximum}$$

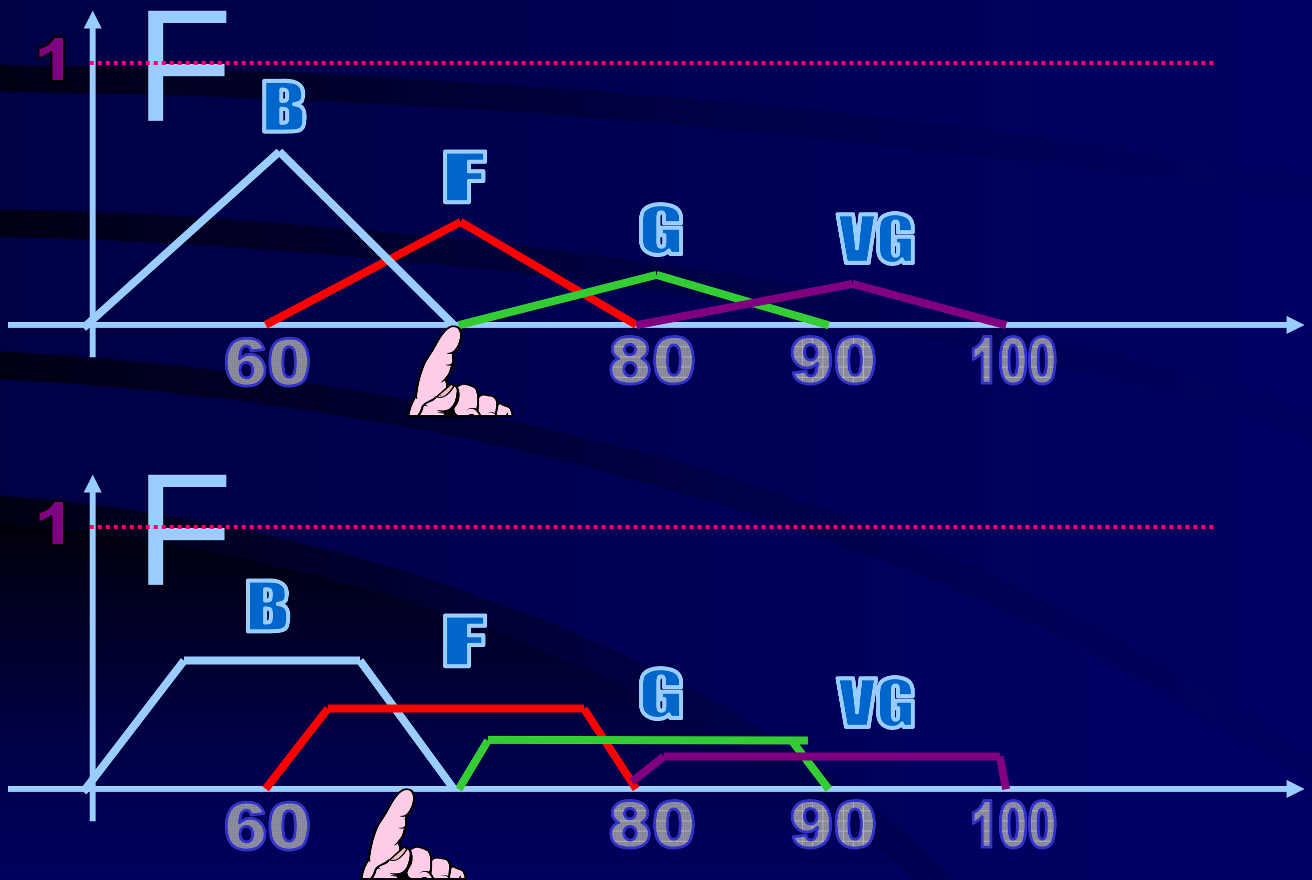
Lab Test: Speed Tracking of IM



Lab Test: Prercision Position Tracking of IM



Commonly Used Variations



Commonly Used Variations

Instead of min(x,y) for fuzzy AND...

Use $\Rightarrow \; x \bullet y$

Instead of max(x,y) for fuzzy OR...

Use $\Rightarrow \; \min(1, x + y)$

Why?

Commonly Used Variations

Sugeno inferencing

Other Norms and co-norms

Relationship with Neural Networks

Explanation Facilities

Teaching a Fuzzy System

Tuning a Fuzzy System